

Phase-noise limitations on single-photon cross-phase modulation with differing group velocities

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A framework is established for evaluating CPHASE gates that use single-photon cross-phase modulation (XPM) originating from the Kerr nonlinearity. Prior work [Phys. Rev. A **73**, 062305 (2006)], which assumed that the control and target pulses propagated at the same group velocity, showed that the causality-induced phase noise required by a non-instantaneous XPM response function precluded the possibility of high-fidelity π -radian conditional phase shifts. The framework presented herein incorporates the more realistic case of group-velocity disparity between the control and target pulses, as employed in existing XPM-based fiber-optical switches. Nevertheless, the causality-induced phase noise identified in [Phys. Rev. A **73**, 062305 (2006)] still rules out high-fidelity π -radian conditional phase shifts. This is shown to be so for both a reasonable theoretical model for the XPM response function and for the experimentally-measured XPM response function of silica-core fiber.

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I. INTRODUCTION

Optics-based quantum computing is an attractive possibility. Single-photon source and detector technologies are rapidly maturing [1–4], enabling robust photonic-qubit creation and detection. Moreover, single-qubit gates are easily realized with linear optics, and photons are the inevitable carriers for the long-distance entanglement distribution needed to network quantum computers. However, optics-based quantum computing is not without its Achilles' heel, namely the extremely challenging task of realizing a high-fidelity, deterministic, two-qubit entangling gate, such as the CPHASE gate.

Knill, Laflamme, and Milburn [5] proposed a solution to the preceding two-qubit gate problem by exploiting the nonlinearity afforded by photodetection in conjunction with the introduction of ancilla photons. Their scheme is intrinsically probabilistic, so it requires high-efficiency adaptive measurement techniques and large quantities of ancilla photons to realize useful levels of quantum computation. Consequently, it remains prudent to continue research on more traditional approaches to all-optical two-qubit gates. A prime example is the nonlinear-optical approach first suggested by Chuang and Yamamoto [6], who proposed using Kerr-effect cross-phase modulation (XPM) to impart a π -radian phase shift on a single-photon pulse, conditioned on the presence of another single-photon pulse.

The fact that the Chuang–Yamamoto architecture provides a deterministic all-optical universal gate set for quantum computation continues to spur work on highly-nonlinear optical fibers [7, 8], but the single-photon level has yet to be reached. Chuang and Yamamoto's analysis treated the control and target as single-spatiotemporal-mode fields. Later work [9], how-

ever, examined their architecture using continuous-time XPM theory. Dismissing the possibility of an instantaneous XPM response—owing to its failure to reproduce experimentally-observed classical results—it showed that a causal, non-instantaneous response function introduces fidelity-degrading phase noise which precludes constructing a high-fidelity CPHASE gate. That analysis assumed control and target pulses propagating at the same group velocity, which implied that a uniform single-photon phase shift could not be realized in the fast-response regime, wherein those pulses have durations much longer than that of the XPM response function. Yet fast-response XPM *is* used for imparting uniform conditional phase shifts in fiber-optical switching with classical control pulses [10]. Those switches' pulses have different group velocities, so that one propagates through the other within the XPM medium.

In this paper we develop a continuous-time quantum XPM theory for pulses with differing group velocities [11], and then use it to assess the feasibility of extending the fiber-switching technique to the single-photon regime for creating a CPHASE gate. We show that causality-induced phase noise still rules out high-fidelity π -radian conditional phase shifts for both a reasonable theoretical model for the XPM response function and for the experimentally-measured XPM response function of silica-core fiber.

II. QUANTUM XPM THEORY

Our theory begins with classical XPM for a pair of single-spatial-mode continuous-time scalar fields—with center frequencies ω_A and ω_B and complex envelopes $E_A(z, t)$ and $E_B(z, t)$ —that propagate from $z = 0$ to $z = L$ through an XPM medium. Because we are interested in ultimate limits on the utility of XPM for two-qubit gates, we neglect loss, dispersion, and self-phase modulation. Thus the behavior of the classical complex

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envelopes of interest is governed by the coupled-mode equations [12]:

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_A} \frac{\partial}{\partial t}\right) E_A(z, t) = \text{in}_A(z, t) E_A(z, t), \quad (1a)$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_B} \frac{\partial}{\partial t}\right) E_B(z, t) = \text{in}_B(z, t) E_B(z, t). \quad (1b)$$

Here, v_A and v_B , satisfying $v_B > v_A$, are the group velocities of $E_A(z, t)$ and $E_B(z, t)$, and $n_A(z, t)$ and $n_B(z, t)$ are the intensity-dependent refractive indices that these fields encounter. For convenient linking to the quantum analysis, we normalize $E_A(z, t)$ and $E_B(z, t)$ to make $\hbar\omega_K I_K(z, t) = \hbar\omega_K |E_K(z, t)|^2$, for $K = A, B$, the powers carried by these fields. The nonlinear refractive indices are then given by [13]

$$n_A(z, t) = \eta \int_{-\infty}^t dt' h(t - t') I_B(z, t'), \quad (2a)$$

$$n_B(z, t) = \eta \int_{-\infty}^t dt' h(t - t') I_A(z, t'), \quad (2b)$$

where η is the strength of the nonlinearity and $h(t)$ is its real-valued, causal response function, normalized to satisfy $\int_0^\infty dt h(t) = 1$.

In the quantum theory for the preceding XPM setup, $E_A(z, t)$ and $E_B(z, t)$ become baseband field operators, $\hat{E}_A(z, t)$ and $\hat{E}_B(z, t)$ with units $\sqrt{\text{photons/s}}$. At the input and output planes, $z = 0$ and $z = L$, these field operators must satisfy the canonical commutation relations for free fields, viz.,

$$[\hat{E}_K(z, t), \hat{E}_J(z, s)] = 0, \quad (3a)$$

$$[\hat{E}_K(z, t), \hat{E}_J^\dagger(z, s)] = \delta_{JK} \delta(t - s), \quad (3b)$$

for $K = A, B$, $J = A, B$, and $z = 0, L$. Unless $h(t) = \delta(t)$, which [9] has ruled out for its failure to reproduce experimentally-observed classical results, Langevin noise terms must be added to the classical coupled-mode equations to ensure that the output fields have the required commutators. Here, we take a cue from the work of Boivin *et al.* [14], which developed a continuous-time quantum theory of self-phase modulation and which [9] extended to XPM when both fields have the same group velocity. The quantum coupled-mode equations that result are

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_A} \frac{\partial}{\partial t}\right) \hat{E}_A(z, t) = i[\hat{n}_A(z, t) + \hat{m}_A(z, t)] \hat{E}_A(z, t), \quad (4a)$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_B} \frac{\partial}{\partial t}\right) \hat{E}_B(z, t) = i[\hat{n}_B(z, t) + \hat{m}_B(z, t)] \hat{E}_B(z, t). \quad (4b)$$

In terms of the photon-flux operators $\hat{I}_K(z, t) \equiv \hat{E}_K^\dagger(z, t) \hat{E}_K(z, t)$, for $K = A, B$, the nonlinear refractive

indices are now operator-valued and given by

$$\hat{n}_A(z, t) = \eta \int_{-\infty}^t dt' h(t - t') \hat{I}_B(z, t'), \quad (5a)$$

$$\hat{n}_B(z, t) = \eta \int_{-\infty}^t dt' h(t - t') \hat{I}_A(z, t'). \quad (5b)$$

The Langevin noise operators $\hat{m}_A(z, t)$ and $\hat{m}_B(z, t)$ are

$$\begin{aligned} \hat{m}_A(z, t) &= \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\eta H_{\text{im}}(\omega)} \\ &\times \{[\hat{B}(z, \omega) - i\hat{C}^\dagger(z, \omega)]e^{-i\omega t} + \text{hc}\}, \end{aligned} \quad (6a)$$

$$\begin{aligned} \hat{m}_B(z, t) &= \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\eta H_{\text{im}}(\omega)} \\ &\times \{[\hat{B}(z, \omega) + i\hat{C}^\dagger(z, \omega)]e^{-i\omega t} + \text{hc}\}, \end{aligned} \quad (6b)$$

where $H_{\text{im}}(\omega)$ is the imaginary part of the frequency response $H(\omega) = \int_0^\infty dt h(t) e^{i\omega t}$, $\hat{B}(z, \omega)$ and $\hat{C}(z, \omega)$ are independent frequency-domain bosonic field operators [15] taken to be in thermal states at absolute temperature T , and hc denotes the Hermitian conjugate.

Equation (4) can be solved to yield the following input-output relations

$$\hat{E}_A^{\text{out}}(t) = e^{i\hat{\xi}_A(t)} e^{i\hat{\zeta}_A(t)} \hat{E}_A^{\text{in}}(t), \quad (7a)$$

$$\hat{E}_B^{\text{out}}(t) = e^{i\hat{\xi}_B(t)} e^{i\hat{\zeta}_B(t)} \hat{E}_B^{\text{in}}(t), \quad (7b)$$

for the output field operators, $\hat{E}_K^{\text{out}}(t) \equiv \hat{E}_K(L, t + L/v_K)$, in terms of the input field operators, $\hat{E}_K^{\text{in}}(t) \equiv \hat{E}_K(0, t)$, the *phase-shift operators* [16]

$$\hat{\zeta}_A(t) \equiv \eta \int_0^L dz \int ds h(t - s) \hat{I}_B^{\text{in}}(s + z/u), \quad (8a)$$

$$\hat{\zeta}_B(t) \equiv \eta \int_0^L dz \int ds h(t - s) \hat{I}_A^{\text{in}}(s - z/u), \quad (8b)$$

where $\hat{I}_K^{\text{in}}(t) \equiv \hat{E}_K^{\text{in}\dagger}(t) \hat{E}_K^{\text{in}}(t)$ and $1/u \equiv 1/v_A - 1/v_B$, and the *phase-noise operators*

$$\begin{aligned} \hat{\xi}_A(t) &\equiv \int_0^L dz \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\eta H_{\text{im}}(\omega)} \\ &\times \{[\hat{B}(z, \omega) - i\hat{C}^\dagger(z, \omega)]e^{-i\omega(t+z/v_A)} + \text{hc}\}, \end{aligned} \quad (9a)$$

$$\begin{aligned} \hat{\xi}_B(t) &\equiv \int_0^L dz \int_0^\infty \frac{d\omega}{2\pi} \sqrt{\eta H_{\text{im}}(\omega)} \\ &\times \{[\hat{B}(z, \omega) + i\hat{C}^\dagger(z, \omega)]e^{-i\omega(t+z/v_B)} + \text{hc}\}. \end{aligned} \quad (9b)$$

These input-output relations ensure that $\hat{E}_A^{\text{out}}(t)$ and $\hat{E}_B^{\text{out}}(t)$ have the proper free-field commutators, as required by Eq. (3).

The phase-noise operators have nonzero commutator

$$\begin{aligned} [\hat{\xi}_A(t), \hat{\xi}_B(s)] &= \\ i\eta \int_0^L dz [h(s - t - z/u) - h(t - s + z/u)], \end{aligned} \quad (10)$$

and they are in a zero-mean jointly Gaussian state that is characterized by the symmetrized autocorrelation functions

$$\langle \hat{\xi}_K(t) \hat{\xi}_K(s) + \hat{\xi}_K(s) \hat{\xi}_K(t) \rangle = \int \frac{d\omega}{\pi} S_{\xi\xi}(\omega) \cos[\omega(t-s)], \quad (11)$$

for $K = A, B$, with spectrum

$$S_{\xi\xi}(\omega) = \eta H_{\text{im}}(\omega) \coth\left(\frac{\hbar\omega}{2k_B T}\right), \quad (12)$$

where k_B is the Boltzmann constant. For the theory to make physical sense, it must be that $H_{\text{im}}(\omega) \geq 0$ for all $\omega \geq 0$ [14, 17], because noise spectra must be nonnegative.

III. XPM-BASED CPHASE GATE

To build a CPHASE gate from the preceding quantum XPM interaction we proceed as follows. Consistent with dual-rail logic [6], the input and output field operators are chosen to be in states in the Hilbert space spanned by their computational basis states, $\{|0\rangle_K, |1\rangle_K : K = A, B\}$. We will take $|0\rangle_K$ to be the vacuum state, and set

$$|1\rangle_K = \int dt \psi_K(t) |t\rangle_K, \quad (13)$$

where the wave functions $\{\psi_K(t) : K = A, B\}$ are normalized ($\int dt |\psi_K(t)|^2 = 1$), and $|t\rangle_K$ is the state of $\hat{E}_K^{\text{in}}(t)$ or $\hat{E}_K^{\text{out}}(t)$ in which there is a single photon at time t and none at all other times. To enforce the interchangeability of the control and target qubits, we take the single-photon pulses in each field to have the same pulse shape. Moreover, because we have assumed $v_B > v_A$, we will assume that $\psi_B(t) = \psi_A(t - t_d)$, where $t_d > 0$ is a delay, specified below, chosen to allow the single-photon excitation in $\hat{E}_B^{\text{in}}(t)$ to propagate through the one in $\hat{E}_A^{\text{in}}(t)$ while both are within the nonlinear medium, thus ensuring each imposes a uniform phase shift on the other.

Sufficient conditions for guaranteeing a uniform phase shift are intuitive and easily derived. Ignoring the phase noise for now, the phase shifts induced on each field by the presence of a single-photon pulse in the other field are found by taking the partial trace of the phase-shift operator for one field with respect to the other:

$${}_B\langle 1 | e^{i\hat{\zeta}_A(t)} | 1 \rangle_B = \int ds e^{i\eta \int_0^L dz h(t-s+z/u)} |\psi_B(s)|^2, \quad (14a)$$

$${}_A\langle 1 | e^{i\hat{\zeta}_B(t)} | 1 \rangle_A = \int ds e^{i\eta \int_0^L dz h(t-s-z/u)} |\psi_A(s)|^2. \quad (14b)$$

From these expressions it is clear that a sufficient condition for a uniform phase shift on $\hat{E}_K^{\text{out}}(t)$ is that the

response-function integrals, i.e., the integrals in the exponents in Eqs. (14a) and (14b), encapsulate the entirety of the response function for all times t and s for which $\psi_K(t)$ and $\psi_J(s)$ are nonzero, where $J \neq K$. Suppose that $h(t)$, $\psi_A(t)$, and $\psi_B(t)$ are only nonzero over the intervals $[0, t_h]$, $[-t_\psi/2, t_\psi/2]$, and $[-t_\psi/2 + t_d, t_\psi/2 + t_d]$, respectively. Although these conditions might not be satisfied exactly, e.g., when the response function and/or the pulse shapes do not have bounded support, we can at least take t_h and t_ψ to represent the nominal durations over which each function is significantly different from zero. In terms of these time durations, our sufficient conditions for uniform phase shifts are

$$t_d \geq t_\psi + t_h \quad (15a)$$

$$\frac{L}{u} \geq t_\psi + t_h + t_d. \quad (15b)$$

Under these conditions, we have that

$$\int_0^L dz h(t-s+z/u) = \int_0^L dz h(t-s-z/u) = u, \quad (16)$$

in Eqs. (14a) and (14b), respectively, due to the normalization of $h(t)$. Hence these equations reduce to

$${}_B\langle 1 | e^{i\hat{\zeta}_A(t)} | 1 \rangle_B = {}_A\langle 1 | e^{i\hat{\zeta}_B(t)} | 1 \rangle_A = e^{i\eta u}, \quad (17)$$

after making use of the wave functions' normalization. From this result we see that the uniform phase shift imposed by the presence of a single-photon excitation is

$$\phi = \eta u, \quad (18)$$

so that the strength, η , of the XPM nonlinearity must be $\eta = \pi/u$ to realize the desired CPHASE gate

Physically, the condition in Eq. (15a) implies that the fast pulse, $\hat{E}_B(z, t)$, does not enter the XPM medium until the entirety of the slow pulse, $\hat{E}_A(z, t)$, and its nonlinear response have propagated into it. Similarly, Eq. (15b) implies that the slow pulse, $\hat{E}_A(z, t)$, does not exit the XPM medium until the entirety of the fast pulse, $\hat{E}_B(z, t)$, and its nonlinear response have propagated out of it. Taking both the delay and the XPM medium to be as short as possible, which will prove most favorable with regards to the phase noise, these conditions simplify to

$$t_d = t_\psi + t_h \quad (19a)$$

$$\frac{L}{u} = 2(t_\psi + t_h). \quad (19b)$$

IV. VACUUM AND SINGLE-PHOTON FIDELITIES

A complete fidelity analysis for the preceding XPM-based CPHASE gate would evaluate the overlap between the actual two-field output state from the XPM interaction and the two-field output state from an ideal CPHASE

gate, averaged uniformly over all possible two-field input states. We, however, will limit our attention to the vacuum and single-photon fidelities, introduced in [9]. Let $\hat{E}_B^{\text{in}}(t)$, regarded as the gate's control field, be in its vacuum (single-photon) state. The ensuing vacuum (single-photon) fidelity F_0 (F_1) is the overlap between the actual state for $\hat{E}_A^{\text{out}}(t)$ and that field's ideal state, averaged uniformly over all $\hat{E}_A^{\text{in}}(t)$ states on the Bloch sphere. The phase-noise limits we will find for these two fidelities are closely related, so let us begin with $\hat{E}_B^{\text{in}}(t)$ being in its vacuum state, in which case the ideal CPHASE-gate state for $\hat{E}_A^{\text{out}}(t)$ is its input state. The formula from [9] for the vacuum fidelity is

$$F_0 = \frac{1}{3} \left[1 + \text{Re} \left(\int dt \langle e^{i\hat{\xi}_A(t)} \rangle |\psi_A(t)|^2 \right) + \int dt \int ds |\psi_A(t)|^2 |\psi_A(s)|^2 \langle e^{i[\hat{\xi}_A(t) - \hat{\xi}_A(s)]} \rangle \right]. \quad (20)$$

We can place an upper bound on the vacuum fidelity by letting $T = 0$ K, to minimize the phase noise, and setting the third term in Eq. (20) to 1, to maximize its value. Using the $\langle e^{i\hat{\xi}_A(t)} \rangle$ value for the phase noise's zero-mean Gaussian state whose spectrum is given by Eq. (12) with $T = 0$ K, we find that

$$F_0 \leq \frac{2}{3} + \frac{1}{3} \exp \left(-\frac{\eta L}{4\pi} \int d\omega |H_{\text{im}}(\omega)| \right). \quad (21)$$

Under our uniform-phase-shift conditions, Eqs. (19) and (18), this bound on the vacuum fidelity becomes

$$F_0 \leq \frac{2}{3} + \frac{1}{3} \exp \left(-\frac{\phi}{2\pi} (t_\psi + t_h) \int d\omega |H_{\text{im}}(\omega)| \right). \quad (22)$$

It is readily apparent from (22) that the vacuum fidelity decreases as the phase shift increases. Likewise, it is clear that perfect fidelity for a nonzero phase shift is impossible, *even in theory*, for any physically-valid response function. Perfect fidelity for a nonzero phase-shift ϕ requires either $t_\psi + t_h = 0$ or $|H_{\text{im}}(\omega)| = 0$ for all ω . The former is impossible for non-instantaneous pulse shapes and response functions, while the latter is impossible for non-instantaneous, causal response functions. An even looser, more favorable bound can be gotten by presuming operation to be in the slow-response regime, wherein $t_\psi \ll t_h$. For a π -radian phase shift, we are then left with

$$F_0 \leq F_{\text{max}} \equiv \frac{2}{3} + \frac{1}{3} \exp \left(-(t_h/2) \int d\omega |H_{\text{im}}(\omega)| \right). \quad (23)$$

This F_{max} result also applies to the single-photon fi-

delity, whose general expression is [9],

$$F_1 = \frac{1}{3} \left[1 + \text{Re} \left(e^{-i\phi} \int dt \langle e^{i\hat{\xi}_A(t)} \rangle \langle e^{i\hat{\zeta}_A(t)} \rangle |\psi_A(t)|^2 \right) + \int dt \int ds |\psi_A(t)|^2 |\psi_A(s)|^2 \langle e^{i[\hat{\xi}_A(t) - \hat{\xi}_A(s)]} \rangle \times \langle e^{i[\hat{\zeta}_A(t) - \hat{\zeta}_A(s)]} \rangle \right], \quad (24)$$

which reduces to the result in Eq. (20) when the XPM interaction produces a uniform ϕ -radian phase shift. Assuming $\phi = \pi$ and operation in the slow-response regime, we then get

$$F_1 \leq \frac{2}{3} + \frac{1}{3} \exp \left(-(t_h/2) \int d\omega |H_{\text{im}}(\omega)| \right) \quad (25)$$

from Eq. (24), thus putting the same optimistic but likely unobtainable upper limit on both the vacuum and single-photon fidelities for an XPM-based gate that produces uniform π -radian phase shifts.

V. PRINCIPAL MODE PROJECTION

The fidelity upper limit we have found for both the vacuum and single-photon fidelities increases with decreasing phase shift, so a natural question arises: Can we cascade a series of small-phase-shift gates, interspersed with quantum error correction, to realize a high-fidelity CPHASE gate? The errors addressed by quantum-computation error correction—dephasing noise, depolarizing noise, bit flips, etc.—all lie *within* the Hilbert space for the qubits of interest [18]. In our case, however, phase noise randomly distorts the single-photon pulse shape while it preserves photon flux, $\hat{I}_K^{\text{in}}(t) = \hat{I}_K^{\text{out}}(t)$, so there is no photon loss. Thus it causes the state to drift *outside* the computational Hilbert space, rendering traditional quantum-error-correction techniques of no value.

An alternative approach would be to reshape the pulses after each XPM interaction, but the random nature of the phase noise precludes this approach's succeeding. Instead, let us pursue the route of principal-mode projection (PMP), as suggested in [19]. There, a V-type atomic system in a one-sided cavity was part of a unit cell comprising the atomic nonlinearity followed by filtering to project its output onto the computational-basis temporal mode (the *principal* mode). Cascading a large number of these unit cells—each producing a small phase shift but with an even smaller error—yielded the π -radian phase shift needed for a CPHASE gate with a fidelity that, in principle, could be arbitrarily high if enough unit cells were employed. It behooves us to see whether a similar favorable error versus phase-shift tradeoff applies to our XPM system. Sadly, as we now show, such is not the case.

Consider a single iteration of XPM+PMP when $\hat{E}_B^{\text{in}}(t)$ is in state $\alpha|0\rangle_A + \beta|1\rangle_A$, with $|\alpha|^2 + |\beta|^2 = 1$, and $\hat{E}_B^{\text{in}}(t)$

is in its vacuum state. The density operator for $\hat{E}_A^{\text{out}}(t)$ will then be

$$\hat{\rho}_{\text{PMP}}^{(0)} = (1 - |\beta|^2 \langle |\hat{T}|^2 \rangle) |0\rangle_A \langle 0| + \alpha\beta^* \langle \hat{T}^\dagger \rangle |0\rangle_A \langle 1| + \alpha^*\beta \langle \hat{T} \rangle |1\rangle_A \langle 0| + |\beta|^2 \langle |\hat{T}|^2 \rangle |1\rangle_A \langle 1|, \quad (26)$$

where $\hat{T} \equiv \int dt |\psi_A(t)|^2 e^{i\hat{\xi}_A(t)}$ can be thought of as the photon-flux transmissivity of the abstract pulse-shape filter responsible for carrying out the PMP. If the XPM interaction produces a uniform ϕ -radian phase shift, then the *same* expression gives the density operator for $\hat{E}_A^{\text{out}}(t)$ when $\hat{E}_B^{\text{in}}(t)$ is in its single-photon state $|1\rangle_B$. Consequently, after averaging $\{\alpha, \beta\}$ over the Bloch sphere we find that the vacuum and single-photon fidelities satisfy

$$F_0 = F_1 = \frac{1}{2} + \frac{1}{3} \langle \text{Re}(\hat{T}) \rangle + \frac{1}{6} \langle |\hat{T}|^2 \rangle \quad (27)$$

$$= \frac{1}{2} + \frac{1}{3} \langle e^{i\hat{\xi}_A(t)} \rangle + \frac{1}{6} \int dt \int ds |\psi_A(t)|^2 |\psi_A(s)|^2 \langle e^{i[\hat{\xi}_A(t) - \hat{\xi}_A(s)]} \rangle, \quad (28)$$

where we have used the fact that $\langle e^{i\hat{\xi}_A(t)} \rangle$ is constant and real-valued. Comparing this result to Eq. (20), we see that a single iteration of PMP *does* increase both F_0 and F_1 , but it does *not* increase F_{max} from what is given in (23), a bound that still applies to both the vacuum and single-photon fidelities.

Now it is easy to see that cascading N unit cells of XPM+PMP cannot avoid the fidelity limit identified in the previous section. For such a cascade, F_0 and F_1 obey Eq. (27) with \hat{T} replaced by $\prod_{n=1}^N \hat{T}_n$, where $\hat{T}_n \equiv \int dt |\psi_A(t)|^2 e^{i\hat{\xi}_{A_n}(t)}$ is the photon-flux transmissivity of the n th XPM+PMP unit cell. But, the $\{\hat{\xi}_{A_n}(t)\}$ are statistically independent and identically distributed, so that Eq. (28) for the N unit-cell cascade is then

$$F_0 = F_1 = \frac{1}{2} + \frac{1}{3} \prod_{n=1}^N \langle e^{i\hat{\xi}_{A_n}(t)} \rangle + \frac{1}{6} \int dt \int ds |\psi_A(t)|^2 |\psi_A(s)|^2 \prod_{n=1}^N \langle e^{i[\hat{\xi}_{A_n}(t) - \hat{\xi}_{A_n}(s)]} \rangle \quad (29)$$

$$\leq \frac{2}{3} + \frac{1}{3} \exp\left(- (t_h/2) \int d\omega |H_{\text{im}}(\omega)|\right), \quad (30)$$

where the inequality is obtained by assuming that each XPM+PMP unit cell operates in the slow-response regime and provides a uniform phase shift of π/N . That this fidelity bound coincides with F_{max} for a single XPM interaction that produces a uniform π -radian phase shift is a consequence of quantum XPM's phase shift and error scaling identically with the nonlinearity's strength, η .

VI. FIBER-XPM FIDELITY BOUNDS

In this section we will evaluate the fidelity bound F_{max} for two XPM response functions: a reasonable theoretical model and the experimentally-measured response function of silica-core fiber. We start with the family of single-resonance, two-pole response functions characterized by the frequency response

$$H(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\gamma}. \quad (31)$$

This family, which was employed in [9], includes a common approximation to the Raman response function of silica-core fiber [20]. For $0 < \gamma/2 < \omega_0$, its response function $h(t)$ is underdamped,

$$h(t) = \frac{\omega_0^2 e^{-\gamma t/2} \sin\left(\sqrt{\omega_0^2 - \gamma^2/4} t\right)}{\sqrt{\omega_0^2 - \gamma^2/4}}, \text{ for } t \geq 0; \quad (32)$$

for $\gamma/2 = \omega_0$ it is critically damped,

$$h(t) = \omega_0^2 t e^{-\omega_0 t}, \text{ for } t \geq 0; \quad (33)$$

and for $\gamma/2 > \omega_0$ it is overdamped,

$$h(t) = \frac{\omega_0^2 e^{-\gamma t/2} \sinh\left(\sqrt{\gamma^2/4 - \omega_0^2} t\right)}{\sqrt{\gamma^2/4 - \omega_0^2}}, \text{ for } t \geq 0. \quad (34)$$

In all of these cases $h(t)$ has infinite duration, so we will optimistically take t_h to be the root-mean-square duration of $h(t)$,

$$t_h = \sqrt{\frac{\int_0^\infty dt t^2 h^2(t)}{\int_0^\infty dt h^2(t)} - \left(\frac{\int_0^\infty dt t h^2(t)}{\int_0^\infty dt h^2(t)}\right)^2} \quad (35)$$

$$= \sqrt{\frac{1}{\gamma^2} + \frac{\gamma^2}{4\omega_0^4} - \frac{1}{2\omega_0^2}}, \quad (36)$$

which satisfies

$$\omega_0 t_h = \sqrt{\frac{1}{\Gamma^2} + \frac{\Gamma^2}{4} - \frac{1}{2}}, \quad (37)$$

in terms of the dimensionless parameter $\Gamma = \gamma/\omega_0$. This duration is minimized at $\Gamma = \sqrt{2}$, which is slightly into the underdamped regime. In terms of Γ , it can be shown that

$$\frac{\int d\omega |H_{\text{im}}(\omega)|}{\omega_0} = \frac{\pi i + 2 \tanh^{-1}\left(\frac{\Gamma^2 - 2}{\Gamma\sqrt{\Gamma^2 - 4}}\right)}{\sqrt{\Gamma^2 - 4}}, \quad (38)$$

which makes it easy to evaluate F_{max} , as a function of Γ , from Eq. (23), as shown in Fig. 1. Note that, despite its appearance, the expression on the right in Eq. (38) is real-valued for $\Gamma \geq 0$.

Figure 1 shows that F_{max} peaks at just less than 82%. It is worth emphasizing, in this regard, that F_{max} is a

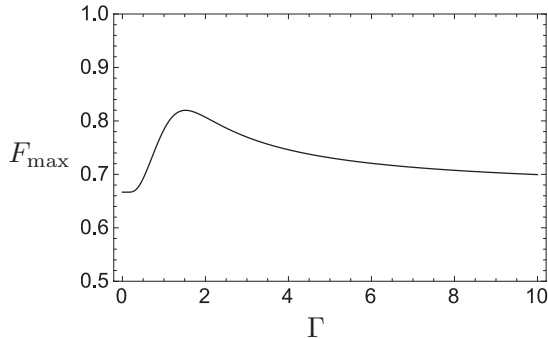


Figure 1: Fidelity upper-bound F_{\max} for the single-resonance, two-pole response function plotted versus the normalized damping parameter Γ .

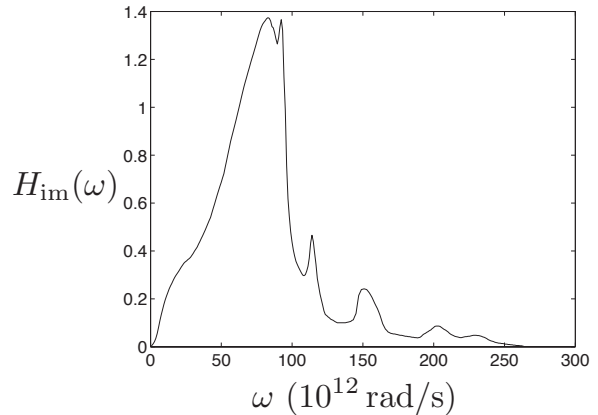
very generous upper bound: (1) it does not include the effects of loss, dispersion, or self-phase modulation; (2) it assumes operation at $T = 0$ K; (3) it assumes operation in the slow-response regime, which would imply $\psi_A(t)$ and $\psi_B(t)$ had sub-femtosecond durations; (4) its use of $h(t)$'s root-mean-square duration for t_h is an optimistic value insofar as uniform phase-shift conditions are concerned; and (5) it has generously set the third term of Eq. (20) to its upper limit of 1. Accordingly, it seems fair to say that, at least for this response function, fiber XPM will not lead to a high-fidelity CPHASE gate.

At this point we could continue by evaluating the behavior of F_{\max} for other idealized theoretical response functions, but it is better to employ the XPM response function of fused-silica fiber. XPM-based fiber-optical switches typically employ co-polarized inputs [10], and for such inputs that response function is the fiber's co-polarized Raman response function [20], which was measured by Stolen *et al.* [21] and is shown in Fig. 2. For this response we have that the root-mean-square duration of $h(t)$ is $t_h \approx 49.2$ fs and

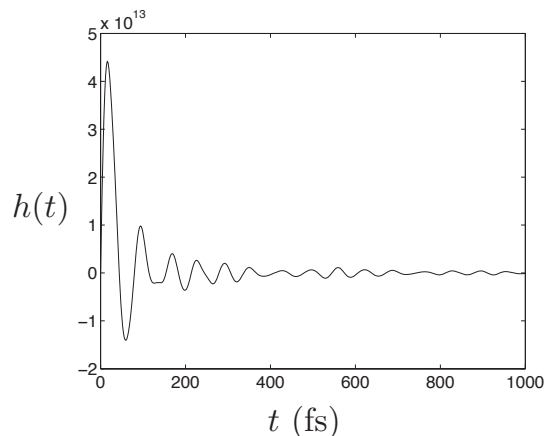
$$\int d\omega |H_{\text{im}}(\omega)| \approx 1.79 \times 10^{14} \text{ rad/s}. \quad (39)$$

These values imply that $F_{\max} \approx 67.1\%$, which is worse than what we found for the Γ -optimized two-pole response.

If we stick with the Raman response function, we can explore F_1 fidelity behavior when we relax our uniform phase-shift condition. In particular, our uniform-phase-shift conditions make good sense when the pulse width is significant relative to the response function's duration. However, deep in the slow-response regime—which these very same conditions suggest is optimal—the $\hat{E}_A^{\text{in}}(t)$ and $\hat{E}_B^{\text{in}}(t)$ pulse shapes are well approximated by Dirac-delta distributions relative to the response function. Thus it would seem that ensuring the entirety of the pulse be exposed to the entirety of the response is not particularly



(a) Frequency response.



(b) Temporal response.

Figure 2: The Raman response of silica-core fiber, as measured by Stolen *et al.* [21].

critical, in this regime, as there is very little pulse to begin with.

Suppose we are aiming for a π -radian phase shift. Then, presuming operation at $T = 0$ K *without* imposing the uniform phase-shift conditions, the vacuum and single-photon fidelities are bounded by (21) for F_0 , and

$$F_1 \leq F_1^{\max} \equiv \frac{2}{3} - \frac{1}{3} \exp\left(-\frac{\eta L}{4\pi} \int d\omega H_{\text{im}}(\omega)\right) \times \text{Re} \left(\int dt \int ds e^{i\eta \int_0^L dz h(t-s+z/u)} |\psi_A(t)|^2 |\psi_B(s)|^2 \right). \quad (40)$$

Decreasing the fiber length, L , at constant nonlinearity strength, η , mitigates the phase-noise fidelity degradation in F_0 . So long as L satisfies the uniform phase-shift condition given in Eq. (19), F_1 will equal

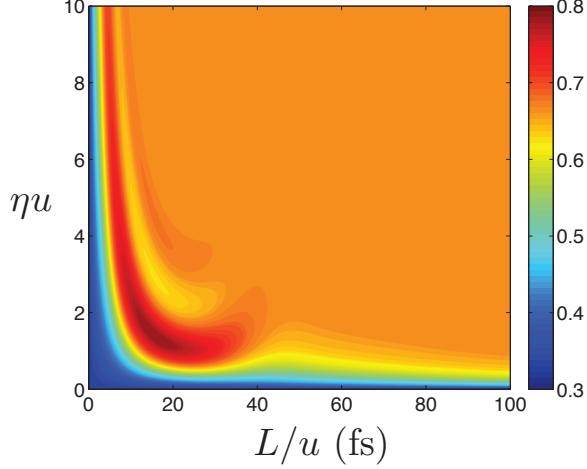


Figure 3: (Color) Heat map of F_1^{\max} versus ηu and L/u for Dirac-delta pulses with $t_d = L/2u$.

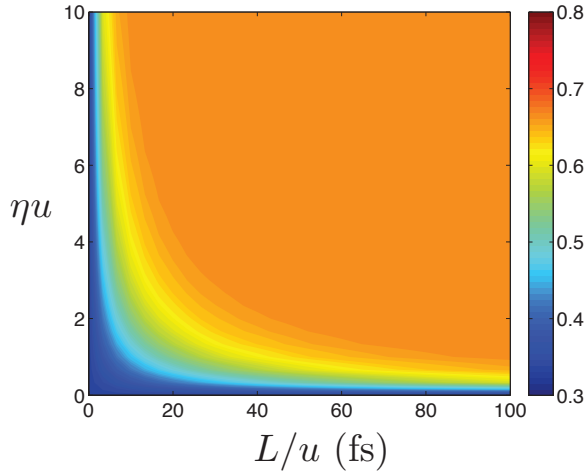


Figure 4: (Color) Heat map of F_1^{\max} versus ηu and L/u for Gaussian pulses with $t_\psi = 3$ ps and $t_d = L/2u$.

F_0 , but once L violates that condition we encounter a tradeoff for F_1^{\max} in Eq. (40): the phase-noise factor, $\exp[-(\eta L/4\pi) \int d\omega H_{\text{im}}(\omega)]$, decreases with further decreases in L , but the factor it multiplies will be greater than the -1 value it had when the phase shift was uniform. In Figs. 3 and 4 we explore that tradeoff.

Figure 3 shows a heat map of F_1^{\max} as ηu and L/u are varied. Here we have assumed the extreme slow-response case of Dirac-delta pulses, and taken $t_d = L/2u$, so that the walk-off between the pulses is symmetric. It turns out that F_1^{\max} peaks at approximately 78.6% when $\eta u \approx$

4.25 and $L/u \approx 16$ fs. Although this peak value exceeds the 67.1% F_{\max} value for fused-silica fiber, it is not very high and is lower than the optimum we gave earlier for the single-resonance, two-pole response function under uniform-phase-shift conditions. Figure 4 shows a similar F_1^{\max} heat map for 3-ps-duration Gaussian pulses, i.e., $\psi_A(t) = e^{-2t^2/t_\psi^2}/(\pi t_\psi^2/4)^{1/4}$ with $t_\psi = 3$ ps. Here we see that the fidelity is abysmal, and our numerical calculation does not yield an $F_1^{\max} > 2/3$. As expected, the uniform-phase-shift conditions are important here—causing the fidelity to be tightly bounded by the phase-noise alone—because operation is well into the fast-response regime.

Taken together, our fidelity bounds for the theoretical and measured response functions permit us to confidently say that XPM in silica-core fiber cannot promise a high-fidelity π -radian CPHASE gate, even under exceedingly idealistic assumptions.

VII. CONCLUSIONS

We have presented a continuous-time, quantum theory for cross-phase modulation with differing group velocities and have provided a framework for evaluating the fidelity of using quantum XPM to construct a CPHASE gate. We found that perfect fidelity is impossible, even in theory, owing to causality-induced phase noise associated with Raman scattering in fused-silica fiber. For a reasonable theoretical response function and the experimentally-measured response function of silica-core fiber, we found that XPM will not support a high-fidelity CPHASE gate, even under a collection of strictly favorable assumptions. In particular, our analysis ignores loss, dispersion, and self-phase modulation. Loss is especially pernicious, considering the length of fused-silica fiber needed for a single-photon pulse to create a π -radian phase shift on another such pulse.

It is worth noting that the silica-core fiber response function we studied is that for co-polarized pulses. The response function for orthogonally-polarized pulses is much faster than—and 1/3 the strength of—its co-polarized counterpart, owing to its being mediated by an electronic interaction, as opposed to the Raman effect that is responsible for co-polarized XPM. We are not aware of any experimental characterization of the co-polarized response function. Nevertheless, the results in this paper suggest that it too will likely lead to low fidelity, so long as it is non-instantaneous, if for no other reason than that its extreme speed will force operation in the less favorable fast-response regime.

Some final comments are now germane with respect to what potential CPHASE gates are *not*, as yet, precluded by our analysis. First, our results do not apply to XPM contained within a larger interaction system, such as a cavity. Some recent results have suggested that cavity-like systems may support a high-fidelity CPHASE gate, despite noise [19, 22]. To date, however, no one has stud-

ied the CPHASE-gate fidelity afforded by cross-Kerr effect XPM within a cavity. Finally, it is unclear to what extent if at all our results apply to dark-state-polariton XPM in electromagnetically-induced transparency (EIT). EIT theories usually assume an instantaneous interaction, which is sometimes taken to be nonlocal [23]. In the physical world, however, a phenomenon is rarely truly instantaneous, regardless of how good an approximation that may be for various working theories. Our work suggests that phase noise may be an issue for EIT if the response function is not truly instantaneous. That aside, recent work has shown that even instantaneous, nonlocal XPM is subject to the same fidelity-degrading phase noise, with limited exceptions [24]. Together with the fact that EIT involves Raman interactions [25], which are ultimately responsible for phase noise in co-polarized fiber XPM, this

suggests that these systems might have to contend with the sort of fidelity issues presented here. Beyond that, other work has quantified additional fidelity-limiting issues, which may be present in continuous-time XPM, that seem likely to affect EIT systems [26–28].

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